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LETTER TO THE EDITOR

Solving formula of the two-dimensional Toda lattice

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Abstract. A solving formula of the two-dimensional Toda lattice is proved, making use of the direct method. This formula is a generalisation and unification of the Bäcklund transformation, the non-linear superposition formula and the Casorati determinant solutions.

It is known that the two-dimensional Toda lattice has Casorati determinant solutions which are generalisations of the usual N-soliton solutions [1]. A linear Bäcklund transformation was found which generates an $(N+1)$-soliton solution from an N-soliton solution [2, 3]. In this letter, we shall give a generalisation and unification of the above results, making use of the direct substitution means using Hirota's bilinear form [4].

We have the following bilinear form for the two-dimensional Toda lattice

$$f_n(\partial_x \partial_s f_n) - (\partial_x f_n)(\partial_s f_n) - (f_{n+1} f_{n-1} - f_n^2) = 0. \tag{1}$$

Now, given that g_n is a solution of (1), f_n^m $(m = 0, 1, \dots)$ satisfy

$$\partial_x f_n^m = \frac{g_n}{g_{n+1}} f_{n+1}^m + \frac{\partial_x g_{n+1}}{g_{n+1}} f_n^m \tag{2}$$

and

$$\partial_s f_n^m = -\frac{g_{n+1}}{g_n} f_{n-1}^m + \frac{\partial_s g_n}{g_n} f_n^m. \tag{3}$$

We define

$$\begin{aligned} r_n^m &= \begin{vmatrix} f_n^0 & f_{n+1}^0 & \cdots & f_{n+m}^0 \\ f_n^1 & f_{n+1}^1 & \cdots & f_{n+m}^1 \\ \vdots & \vdots & & \vdots \\ f_n^m & f_{n+1}^m & \cdots & f_{n+m}^m \end{vmatrix} (g_{n+1} \cdots g_{n+m})^{-1} \\ &= |n, n+1, \dots, n+m| (g_{n+1} \cdots g_{n+m})^{-1} \end{aligned} \tag{4}$$

where the notation $|n_1, n_2, \dots, n_k|$ is as used in [4]. Using (2) and (3), we have

$$\partial_x r_n^m = \frac{|n, \dots, n+m-1, n+m+1|}{g_{n+1} \cdots g_{n+m-1} g_{n+m+1}} + \frac{(\partial_x g_{n+m+1}) |n, \dots, n+m|}{g_{n+1} \cdots g_{n+m+1}} \tag{5}$$

$$\partial_s r_n^m = -\frac{|n-1, n+1, \dots, n+m|}{g_n g_{n+2} \cdots g_{n+m}} + \frac{(\partial_s g_n) |n, \dots, n+m|}{g_n \cdots g_{n+m}} \tag{6}$$

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$$\begin{aligned}
\partial_x\partial_s r_n^m = \frac{1}{g_n\cdots g_{n+m}}\bigg(&\frac{g_n(\partial_x\partial_s g_n)-(\partial_x g_n)(\partial_s g_n)-g_{n+1}g_{n-1}}{g_n}|n,\ldots,n+m| \\
&+\frac{g_{n+1}g_{n+m}}{g_{n+m+1}}|n-1,n+1,\ldots,n+m-1,n+m+1| \\
&-\frac{g_{n+1}(\partial_x g_{n+m+1})}{g_{n+m+1}}|n-1,n+1,\ldots,n+m| \\
&+\frac{(\partial_s g_n)(\partial_x g_{n+m+1})}{g_{n+m+1}}|n,\ldots,n+m|\bigg)
\end{aligned} \tag{7}$$

$$\begin{aligned}
(\partial_x r_n^m)(\partial_s r_n^m) = \frac{|n,\ldots,n+m|}{g_n g_{n+1}^2\cdots g_{n+m}^2 g_{n+m+1}}&[|n,\ldots,n+m-1,n+m+1|g_{n+m}(\partial_s g_n) \\
&+(\partial_s g_n)(\partial_x g_{n+m+1})|n,\ldots,n+m| \\
&-g_{n+1}(\partial_x g_{n+m+1})|n-1,n+1,\ldots,n+m|] \\
&-\frac{|n-1,n+1,\ldots,n+m|}{g_n\cdots g_{n+m-1}}\frac{|n,\ldots,n+m-1,n+m+1|}{g_{n+2}\cdots g_{n+m+1}}
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&r_n^m(\partial_x\partial_s r_n^m)-(\partial_x r_n^m)(\partial_s r_n^m)-r_{n+1}^m r_{n-1}^m+r_n^m r_n^m \\
&\quad=\left(\frac{|n,\ldots,n+m|}{g_n\cdots g_{n+m}}\right)^2[g_n(\partial_x\partial_s g_n)-(\partial_x g_n)(\partial_s g_n)-g_{n+1}g_{n-1}+g_n^2] \\
&\qquad+\frac{1}{g_{n+2}\cdots g_{n+m+1}}\frac{1}{g_n\cdots g_{n+m-1}} \\
&\qquad\times(|n,\ldots,n+m||n-1,n+1,\ldots,n+m-1,n+m+1| \\
&\qquad-|n-1,n+1,\ldots,n+m||n,\ldots,n+m-1,n+m+1| \\
&\qquad-|n+1,\ldots,n+m+1||n-1,\ldots,n+m-1|) \\
&\quad=-\frac{1}{g_{n+2}\cdots g_{n+m+1}}\frac{1}{g_n\cdots g_{n+m-1}} \\
&\qquad\times\begin{vmatrix} n-1 & n & n+1,\ldots,n+m-1 & 0 & n+m & n+m+1 \\ n-1 & n & 0 & n+1,\ldots,n+m-1 & n+m & n+m+1 \end{vmatrix} \\
&\quad=0.
\end{aligned} \tag{9}$$

Thus r_n^m $(m=0,1,\ldots)$ are solutions of (1).

If we take $g_n=1$, then we get the Casorati determinant solution [1]

$$r_n^m=\begin{vmatrix} f_n^0 & \cdots & f_{n+m}^0 \\ \vdots & \ddots & \vdots \\ f_n^m & \cdots & f_{n+m}^m \end{vmatrix}$$

where $\partial_x f_n^m=f_{n+1}^m$ and $\partial_s f_n^m=-f_{n-1}^m$.

If we take $m=0$, then we know f_n are solutions of (1) [2].
If we take $m=1$, then we get the non-linear superposition formula [5]

$$g_n r_n^1 = \exp(D_n) f_n^1 f_n^0$$

where D_n is Hirota's bilinear difference operator.

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